Iranian Mathematical Society

First Day

40th Iranian Mathematics Competition

24 August 2016

Iran University of Science & Technology

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1. Let $R$ be a ring and $e, f \in R$. Suppose that $e$, $f$, and $e+f$ are idempotents, prove that $ef$ is an idempotent too. (An element $x \in R$ is called idempotent if $x^2 = x$).

2. Show that the function $d : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ defined by $d(m,n) = \log \frac{\sqrt{mn}}{(m,n)}$ is a metric on $\mathbb{N}$. ($(m,n)$ is the greatest common divisor of $m$ and $n$).

3. Let $u_n$ be the unique positive root of the equation $x^n + x^{n-1} + \cdots + x - 1 = 0$. Prove that $\{u_n\}$ is convergent and find its limit.

4. Let $R$ be a finite ring with identity and $U(R)$ be the set of all invertible elements of $R$. Show that $R$ has not any non-zero nilpotent element, provided that $(|U(R)|, |R|) = 1$.

   (We say $x \in R$ is nilpotent if $x^n = 0$ for some $n \in \mathbb{N}$).

5. Let $P_1, \ldots, P_n$ be some points inside a circle of unit radii such that for each point $P$ on the circle, the product $|\overline{PP_1}|.|\overline{PP_2}|.\ldots.|\overline{PP_n}| \leq 1$. Show that all of the points $P_1, \ldots, P_n$ coincide with the center of the circle.

6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function such that its restriction to any line in $\mathbb{R}^2$ is monotone, i.e., the mapping $g : \mathbb{R} \to \mathbb{R}$ defined by $g(t) = f(ta+b)$, is monotone for all $a, b \in \mathbb{R}^2$. Prove that there exist a function $h : \mathbb{R} \to \mathbb{R}$ and a vector $v \in \mathbb{R}^2$ such that $f(x) = h(x.v)$.

   (Here, $x.v$ denotes the inner product of the vectors $x$ and $v$).